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Zeros and Critical Points of Sobolev Orthogonal Polynomials

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Using potential theoretic methods we study the asymptotic distribution of zeros and critical points of Sobolev orthogonal polynomials, i.e., polynomials orthogonal with respect to an inner product involving derivatives. Under general assumptions it is shown that the critical points have a canonical asymptotic limit distribution supported on the real line. In certain cases the zeros themselves have the same asymptotic limit distribution, while in other cases we can only ascertain that the support of a limit distribution lies within a specified set in the complex plane. One of our tools, which is of independent interest, is a new result on zero distributions of asymptotically extremal polynomials. Our results are illustrated by numerical computations for the case of two disjoint intervals. We also describe the numerical methods that were used. © 1997 Academic Press

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We consider a Sobolev inner product

$$\langle f, g \rangle = \int f(t)\, g(t)\, d\mu_0(t) + \int f'(t)\, g'(t)\, d\mu_1(t), \tag{1.1}$$

where μ_0 and μ_1 are compactly supported positive measures on the real line with finite total mass. We put

$$\Sigma_0 := \operatorname{supp}(\mu_0), \qquad \Sigma_1 := \operatorname{supp}(\mu_1), \qquad \Sigma := \Sigma_0 \cup \Sigma_1. \tag{1.2}$$

If, as we assume, μ_0 has infinite support, there exists a unique sequence of monic polynomials π_n, $\deg \pi_n = n$, which is orthogonal with respect to the

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inner product (1.1). These Sobolev orthogonal polynomials have properties that clearly distinguish them from ordinary orthogonal polynomials, most notably by the fact that some or many of the zeros of π_n may be outside the convex hull of Σ, or even off the real line; cf. [1, 9]. In recent papers many results on zeros of special classes of Sobolev orthogonal polynomials were obtained. We refer to the surveys [8, 10].

Asymptotic properties of Sobolev orthogonal polynomials were obtained by López, Marcellán, and Van Assche [7]. These authors considered a general class of inner products, including inner products (1.1) with discrete measure μ_1.

In the present paper, we study the asymptotic behavior of zeros and critical points of orthogonal polynomials in a continuous Sobolev space, i.e., when both μ_0 and μ_1 are nondiscrete measures. Our results will be stated in terms of weak* convergence of measures. We associate with a polynomial P of exact degree n its normalized zero distribution,

$$\nu(P) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j}, \tag{1.3}$$

where $z_1, ..., z_n$ are the zeros of P counted according to their multiplicities. A sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, $\deg P_n = n$, is said to have asymptotic zero distribution μ if μ is a probability measure on $\overline{\mathbf{C}}$ and

$$\lim_{n \to \infty} \int f \, d\nu(P_n) = \int f \, d\mu \tag{1.4}$$

for every continuous function f on $\overline{\mathbf{C}}$. That is, their normalized zero distributions converge in the weak* sense to μ.

Asymptotic zero distributions for orthogonal polynomials with respect to an ordinary inner product

$$\langle f, g \rangle = \int f(t) \, g(t) \, d\mu(t), \qquad \Sigma := \operatorname{supp}(\mu) \subset \mathbf{R}, \tag{1.5}$$

have been studied by many authors. The most comprehensive account can be found in the monograph of Stahl and Totik [13]. They introduce a class **Reg** of regular measures. One of their results is that for $\mu \in \mathbf{Reg}$, the orthogonal polynomials p_n for the inner product (1.5) have regular asymptotic zero distribution. This means that

$$\lim_{n \to \infty} \nu(p_n) = \omega_{\Sigma},$$

where ω_Σ is the equilibrium measure of Σ; see [13, Theorem 3.6.1]. In case $\Sigma = \mathrm{supp}(\mu)$ is regular with respect to the Dirichlet problem in $\mathbf{C}\backslash\Sigma$, the measure μ belongs to **Reg** if and only if

$$\lim_{n\to\infty}\left(\frac{\|P_n\|_\Sigma}{\|P_n\|_{L^2(\mu)}}\right)^{1/n} = 1 \tag{1.6}$$

for every sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, $\deg P_n \leqslant n$, $P_n \not\equiv 0$. Here and in the following we use $\|\cdot\|_\Sigma$ to denote the supremum norm on Σ. Regularity of a measure indicates that it is sufficiently dense on its support. For example, it is enough that μ has a density which is positive almost everywhere on Σ. See [13, Chap. 4] for this and other criteria for regularity of μ.

Motivated by these facts, we make the following assumptions on the measures μ_0 and μ_1 in (1.1). Recall that $\Sigma_j = \mathrm{supp}(\mu_j)$, $j = 0, 1$.

Assumption A. For $j = 0, 1$, the set Σ_j is compact and regular for the Dirichlet problem in $\overline{\mathbf{C}}\backslash\Sigma_j$.

Assumption B. The measures μ_0 and μ_1 belong to the class **Reg**.

Our first result concerns the asymptotic zero distribution for the derivatives π_n' of the Sobolev orthogonal polynomials.

THEOREM 1. *Let μ_0 and μ_1 be measures on the real line satisfying Assumptions* A *and* B. *Let $\{\pi_n\}$ be the sequence of monic orthogonal polynomials for the inner product* (1.1). *Then*

$$\lim_{n\to\infty} \nu(\pi_n') = \omega_\Sigma,$$

where $\Sigma = \mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1)$ and ω_Σ is the equilibrium measure of Σ.

Thus the sequence of derivatives $\{\pi_n'\}$ has regular asymptotic zero distribution. Note, however, that this does not imply that the zeros of π_n' are all real. In fact, we do not even know if the zeros remain uniformly bounded. In our computations we found in all cases that the zeros of π_n' are real, see Section 2. While we have no reason to believe that this is true in general, we feel confident about the following conjecture.

Conjecture 1. Under the same conditions as in Theorem 1, let U be an arbitrary open set containing the convex hull of Σ. Then there is an n_0 such that for every $n \geqslant n_0$, all zeros of π_n' are in U.

To discuss the zeros of the Sobolev orthogonal polynomials π_n themselves, we need to introduce some more notation. Set

$$\Omega := \overline{\mathbf{C}} \backslash \Sigma,$$

and let $g_\Omega(z; \infty)$ be the Green function for Ω with pole at infinity; see [12, 13]. For $r > 0$, we denote by V_r the union of those components of $\{z \in \mathbf{C}\colon g_\Omega(z; \infty) < r\}$ having empty intersection with Σ_0, and we put

$$V := \bigcup_{r>0} V_r.$$

Finally, we put

$$K := \partial V \cup (\Sigma \backslash V).$$

THEOREM 2. *Let μ_0 and μ_1 be measures on the real line satisfying Assumptions* A *and* B. *Let $\{\pi_n\}$ be the sequence of monic orthogonal polynomials for the inner product* (1.1). *Let ν be a weak* limit of a subsequence of $\{\nu(\pi_n)\}$. Then*

(a) $\operatorname{supp}(\nu) \subset \overline{V} \cup \Sigma$,

(b) *the balayage of ν onto K is equal to the balayage of ω_Σ onto K.*

See [13] for the notion of balayage of a measure onto a compact set.

The information on the zeros of π_n we get from Theorem 2 is less precise than the information on the critical points from Theorem 1. In particular, it does not follow that the full sequence $\{\nu(\pi_n)\}$ converges. However, in some cases we can say more.

COROLLARY 3. *Under the same conditions as in Theorem* 2, *let ν be a weak* limit of a subsequence of $\{\nu(\pi_n)\}$. If $K = \Sigma$ (e.g., if $\Sigma_1 \subseteq \Sigma_0$), then $\nu = \omega_\Sigma$. In this case the full sequence $\{\nu(\pi_n)\}$ converges to ω_Σ.*

Corollary 3 follows immediately from Theorem 2. In our numerical examples, see Sections 2.3–2.4, we found that for n up to 50, part of the zeros of π_n are still pretty far outside K. But we conjecture that they do not accumulate outside of $\overline{V}$ and the convex hull of Σ.

Conjecture 2. Under the same conditions as in Theorem 2, let U be an arbitrary open set containing $\overline{V}$ and the convex hull of Σ. Then there is an n_0 such that for every $n \geqslant n_0$, all zeros of π_n are in U.

Conjecture 2 actually follows from Conjecture 1.

The rest of this paper is organized as follows. We first present numerical results on zeros and critical points for several special cases, where Σ consists of two disjoint intervals. The numerical methods we used are discussed in Section 6. The proofs of the theorems are in Sections 3–5. They depend essentially on results on zero distributions of asymptotically minimal polynomials obtained by Blatt, Saff, and Simkani [2] and Mhaskar and Saff [11]. For the proof of Theorem 2 we need an extension of these results, which will be presented as Theorem 5 in Section 3. In Section 4 we give the proof of Theorem 1 and in Section 5 the proof of Theorem 2.

2. TWO DISJOINT INTERVALS: NUMERICAL RESULTS

In this section we present numerical calculations to illustrate our results. The methods used are described in Section 6.

We consider the case where Σ consists of two disjoint intervals of equal length. We choose

$$\Sigma = [-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1].$$

With λ_+ the Lebesgue measure restricted to $[\frac{1}{2}, 1]$ and λ_- the Lebesgue measure restricted to $[-1, -\frac{1}{2}]$, we distinguish the following four cases:

Case A: $\mu_0 = \mu_1 = \lambda_+ + \lambda_-$;

Case B: $\mu_0 = \lambda_+ + \lambda_-$, $\mu_1 = \lambda_-$;

Case C: $\mu_0 = \lambda_+$, $\mu_1 = \lambda_+ + \lambda_-$;

Case D: $\mu_0 = \lambda_+$, $\mu_1 = \lambda_-$.

In all four cases, we know from Theorem 1 that the asymptotic zero distribution for the derivatives is equal to ω_Σ. In Cases A and B we have $\Sigma_1 \subseteq \Sigma_0$. Thus, it follows from Corollary 3 that in these two cases the asymptotic zero distribution for the Sobolev orthogonal polynomials is also equal to ω_Σ. This is confirmed by our calculations.

2.1. *Case A*: $\mu_0 = \mu_1 = \lambda_+ + \lambda_-$ (*Table I*)

In our calculations for $n = 1(1)25(5)50$ we found complex zeros of π_n only for $n = 5$, 7, and 9. All zeros of π_n' were found to be simple, real, and in the interval $(-1, 1)$.

2.2. *Case B*: $\mu_0 = \lambda_+ + \lambda_-$, $\mu_1 = \lambda_-$ (*Table II*)

Again, most of the zeros are real. Only for $n = 4$ and 6 did we find complex zeros of π_n. The zeros of π_n' are all simple, real, and in $(-1, 1)$. (Calculations for the same n as in Case A.)

The situation is different in Cases C and D. In these cases the set K of Theorem 2 may be described as follows. The Green function $g_\Omega(z; \infty)$ of $\Omega = \overline{\mathbb{C}} \backslash \Sigma$ has one level set $\{z: g_\Omega(z; \infty) = r_c\}$ consisting of a figure eight.

TABLE I

Zeros of π_n and π'_n, $n=5$, 10, in Case A

	Zeros of π_n	Zeros of π'_n
$n=5$	$-0.93646854 - 0.20876772i$	-0.88534979
	$-0.93646854 + 0.20876772i$	-0.46499783
	0.0	0.46499783
	$0.93646854 - 0.20876772i$	0.88534979
	$0.93646854 + 0.20876772i$	
$n=10$	-1.00052723	-0.97497028
	-0.93567713	-0.87345927
	-0.80269592	-0.71474572
	-0.62612019	-0.55444777
	-0.50181795	0.0
	0.50181795	0.55444777
	0.62612019	0.71474572
	0.80269592	0.87345927
	0.93567713	0.97497028
	1.00052723	

TABLE II

Zeros of π_n and π'_n, $n=5$, 10, in Case B

	Zeros of π_n	Zeros of π'_n
$n=5$	-1.01982013	-0.91709404
	-0.74396812	-0.64370369
	-0.55435292	0.14139821
	0.61214903	0.78137665
	0.90846355	
$n=10$	-1.00290062	-0.97911875
	-0.93891943	-0.89422735
	-0.84280403	-0.75923516
	-0.66396367	-0.61066220
	-0.55481204	-0.51231989
	-0.48324766	0.16014304
	0.55639877	0.62971341
	0.71942191	0.80459125
	0.87676555	0.93865007
	0.97576614	

For symmetry reasons, this is the level set containing 0. The set K consists of two parts. It is the union of $[\frac{1}{2}, 1]$ with that part of the figure eight that encircles $[-1, -\frac{1}{2}]$.

2.3. *Case C*: $\mu_0 = \lambda_+$, $\mu_1 = \lambda_+ + \lambda_-$ (*Table III*)

In our calculations for $n = 1(1)25(5)50$ all zeros of π'_n were found to be simple, real, and in $(-1, 1)$. All zeros of π_n are real only for $n = 1, 2, 3, 4, 6, 8$, and 10. All complex zeros have a negative real part and they are encircling $[-1, -\frac{1}{2}]$. Furthermore, we noted some peculiarities in the behavior of the complex zeros. For odd n, the complex zeros are outside

TABLE III

Zeros of π_n and π'_n, $n = 5, 10, 15$, in Case C

	Zeros of π_n	Zeros of π'_n
$n=5$	$-1.13970225 - 0.44661459i$	-0.90932823
	$-1.13970225 + 0.44661459i$	-0.62403037
	0.50779290	0.62478703
	0.76816794	0.90887919
	1.00382819	
$n=10$	-0.98774277	-0.97498555
	-0.95967689	-0.87349586
	-0.77454092	-0.71478191
	-0.65462781	-0.55436421
	-0.48961896	0.00056691
	0.50181827	0.55445253
	0.62612626	0.71475358
	0.80270124	0.87346371
	0.93567933	0.97497123
	1.00052715	
$n=15$	-1.20729028	-0.99008732
	$-1.11842498 - 0.23762201i$	-0.94869995
	$-1.11842498 + 0.23762201i$	-0.87812479
	$-0.86567461 - 0.41291713i$	-0.78542939
	$-0.86567461 + 0.41291713i$	-0.68199701
	$-0.48045299 - 0.45544118i$	-0.58497964
	$-0.48045299 + 0.45544118i$	-0.51762420
	0.50000295	0.51762967
	0.54387032	0.58499199
	0.63049097	0.68200314
	0.73428763	0.78542581
	0.83481287	0.87811753
	0.91801959	0.94869496
	0.97492010	0.99008612
	0.99999844	

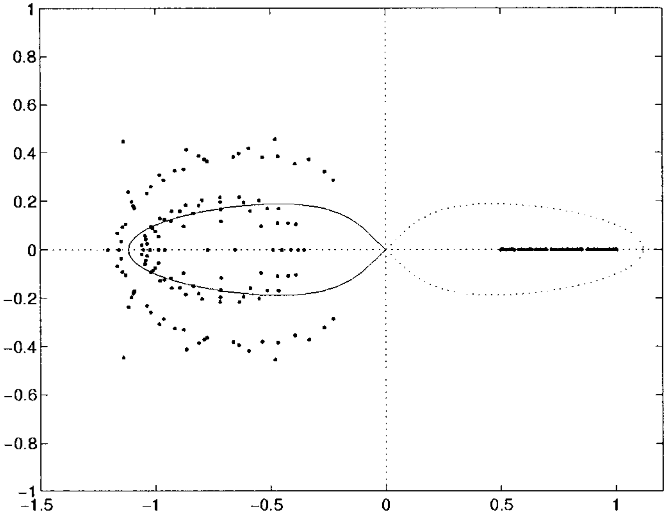

FIG. 1. Plot of the zeros of π_n, $n = 5(5)50$, in Case C.

the set K, while for even n, they are initially inside, but eventually some cross over to the outside. It seems likely that for odd n, the zeros tend to K from the outside but the convergence is very slow. For even n, there might be a different limit distribution, although it is conceivable that also for even n, the zeros accumulate on K. It is also remarkable that the zeros

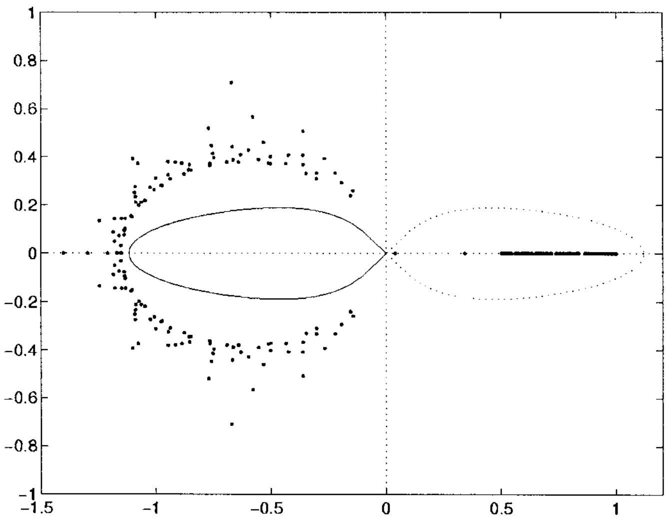

FIG. 2. Plot of the zeros of π_n, $n = 5(5)50$, in Case D.

of π_n' are very close to being symmetric around 0. We have no explanations for these phenomena.

Figure 1 depicts the zeros of π_n, $n=5(5)50$, along with that part of K that encircles $[-1, -\frac{1}{2}]$.

2.4. *Case D*: $\mu_0=\lambda_+$, $\mu_1=\lambda_-$ (*Table IV*)

We found complex zeros of π_n for all n, except $n=1$, 2, and 3. Again, all the zeros of π_n' are simple, real, and in $(-1, 1)$.

In contrast to Case C, we found no zeros of π_n inside the curve K (except for $n=3$). This is illustrated in Fig. 2 with the plots of the zeros of π_n, $n=5(5)50$. Note that the zeros are pretty far from K.

TABLE IV

Zeros of π_n and π_n', $n=5$, 10, 15, in Case D

	Zeros of π_n			Zeros of π_n'
$n=5$	−1.40237979			−0.91931357
	−0.67193855	−	0.70835815i	−0.64605904
	−0.67193855	+	0.70835815i	−0.18436141
	0.62935932			0.78712860
	0.91364079			
$n=10$	−1.29703537			−0.98088476
	−1.10126374	−	0.39294199i	−0.90316848
	−1.10126374	+	0.39294199i	−0.77960092
	−0.57893971	−	0.56595190i	−0.63989830
	−0.57893971	+	0.56595190i	−0.53049964
	0.51468739			0.55298588
	0.60589851			0.68147141
	0.75300437			0.83024743
	0.89081502			0.94619968
	0.97842844			
$n=15$	−1.24663987	−	0.13488685i	−0.99138203
	−1.24663987	+	0.13488685i	−0.95536746
	−1.07914072	−	0.37346724i	−0.89378004
	−1.07914072	+	0.37346724i	−0.81229432
	−0.77108509	−	0.51962021i	−0.71962499
	−0.77108509	+	0.51962021i	−0.62805945
	−0.36124445	−	0.50773392i	−0.55324965
	−0.36124445	+	0.50773392i	−0.50975247
	0.51791298			0.54446702
	0.58620377			0.63199538
	0.68402014			0.73630329
	0.78755144			0.83660513
	0.87969723			0.91913536
	0.94947423			0.97530045
	0.99024926			

2.5. *Another Choice for λ_+ and λ_-*

We also experimented with λ_+ the measure $|t|\,(t^2-\frac{1}{4})^{-1/2}\,(1-t^2)^{-1/2}$ restricted to $[\frac{1}{2}, 1]$ and λ_- the same measure restricted to $[-1, -\frac{1}{2}]$. The results, on the whole, are very similar to those for the Lebesgue measure. The differences noted were that complex zeros of π_n occur also for $n=11$ and 13 in Case A, and for $n=8$ in Case B. In Case C, all zeros of π_n are real only for $n=1$, 2, 3, 4, 6, and 8.

3. AN AUXILIARY RESULT ON ASYMPTOTICALLY MINIMAL POLYNOMIALS

A major tool in the proof of Theorem 1 is a well-known result on zero distributions of polynomials, which we state below for the case of a set $E\subset\mathbf{R}$. Here and in the following, cap(E) denotes the logarithmic capacity of E; see, e.g., [12, 13].

LEMMA 4. *Let $E\subset\mathbf{R}$ be compact with* $\text{cap}(E)>0$ *and let $\{p_n\}$ be a sequence of monic polynomials,* $\deg p_n=n$, *such that*

$$\limsup_{n\to\infty} \|p_n\|_E^{1/n} \leqslant \text{cap}(E). \tag{3.1}$$

Then

$$\lim_{n\to\infty} \nu(p_n)=\omega_E. \tag{3.2}$$

Proof. See the paper of Blatt, Saff, and Simkani [2]. ∎

Monic polynomials satisfying (3.1) are called asymptotically minimal polynomials, since every monic polynomial p_n of degree n satisfies

$$\|p_n\|_E^{1/n} \geqslant \text{cap}(E).$$

Hence, if (3.1) holds, we have in fact equality.

A weighted analogue of this theorem was obtained by Mhaskar and Saff [11]. To prove Theorem 2, we will need a slightly stronger result, which may be of independent interest. To state it, we recall the situation of [11]. Assume $E\subset\mathbf{C}$ is a closed set. A function $w\colon E\to[0,\infty)$ is an admissible weight if

(a) w is upper semicontinuous;

(b) the set $\{z\in E\colon w(z)>0\}$ has positive capacity;

(c) if E is unbounded, then $|z|\,w(z)\to 0$ as $|z|\to\infty$, $z\in E$.

Associated with an admissible weight w is a unique positive unit measure μ_w and a unique constant F_w such that

$$\begin{aligned} U^{\mu_w}(z) - \log w(z) &= F_w \qquad \text{q.e. on } \operatorname{supp}(\mu_w), \\ U^{\mu_w}(z) - \log w(z) &\geqslant F_w \qquad \text{q.e. on } E. \end{aligned} \tag{3.3}$$

Here, U^μ denotes the logarithmic potential of the measure μ,

$$U^\mu(z) := \int \log \frac{1}{|z-t|} \, d\mu(t),$$

and q.e. means quasi-everywhere, that is, except for a set of zero capacity.

In the following theorem we use S_w to denote the support of μ_w, $Pc(S_w)$ denotes the polynomial convex hull of S_w, $D_w = \overline{\mathbf{C}} \backslash Pc(S_w)$ denotes the unbounded component of $\overline{\mathbf{C}} \backslash S_w$, and ∂D_w denotes the boundary of D_w (also known as the outer boundary of S_w).

THEOREM 5. *Let w be an admissible weight on the closed set $E \subset \mathbf{C}$. Let $\{p_n\}_{n=1}^\infty$ be a sequence of monic polynomials*, $\deg p_n = n$, *such that for q.e. $z \in \partial D_w$,*

$$\limsup_{n \to \infty} \left[w(z) \, |p_n(z)|^{1/n} \right] \leqslant \exp(-F_w). \tag{3.4}$$

Then for every closed $A \subset D_w$,

$$\lim_{n \to \infty} \nu(p_n)(A) = 0. \tag{3.5}$$

Furthermore, if ν is the weak limit of a subsequence of $\{\nu(p_n)\}$, then* $\operatorname{supp}(\nu^*) \subset Pc(S_w)$ *and the balayage of ν^* onto ∂D_w is equal to the balayage of μ_w onto ∂D_w.*

In [11] the same result was obtained from the stronger assumption

$$\limsup_{n \to \infty} \|w^n p_n\|_{\partial D_w}^{1/n} \leqslant \exp(-F_w).$$

Proof. In terms of potentials, the relation (3.4) is

$$F_w + \log w(z) \leqslant \liminf_{n \to \infty} U^{\nu(p_n)}(z), \qquad \text{q.e. } z \in \partial D_w,$$

and in view of (3.3) this implies

$$U^{\mu_w}(z) \leqslant \liminf_{n \to \infty} U^{\nu(p_n)}(z), \qquad \text{q.e. } z \in \partial D_w. \tag{3.6}$$

Let ν_n be the balayage of $\nu(p_n)$ onto $Pc(S_w)$. Then

$$U^{\nu_n}(z) = U^{\nu(p_n)}(z) + c_n, \qquad \text{q.e. } z \in Pc(S_w), \tag{3.7}$$

with a constant c_n given by (see [13, Appendix VII])

$$c_n = \int g_{D_w}(z; \infty)\, d\nu(p_n)(z) \geqslant 0. \tag{3.8}$$

Let ν be the weak* limit of a subsequence of $\{\nu_n\}$, say $\nu_n \to \nu$ as $n \to \infty$, $n \in \Lambda$, where Λ is a subsequence of the natural numbers. Then $\operatorname{supp}(\nu) \subset Pc(S_w)$, and by the lower envelope theorem [13, Appendix III]

$$U^{\nu}(z) = \liminf_{n \to \infty,\, n \in \Lambda} U^{\nu_n}(z), \qquad \text{q.e. } z \in \mathbf{C}.$$

Combining this with (3.7), (3.8), and (3.6), we find for q.e. $z \in \partial D_w$:

$$\begin{aligned} U^{\nu}(z) &= \liminf_{n \to \infty,\, n \in \Lambda} U^{\nu_n}(z) = \liminf_{n \to \infty,\, n \in \Lambda} \left[U^{\nu(p_n)}(z) + c_n \right] \\ &\geqslant \liminf_{n \to \infty,\, n \in \Lambda} U^{\nu(p_n)}(z) \geqslant U^{\mu_w}(z). \end{aligned} \tag{3.9}$$

Since $U^{\nu} - U^{\mu_w}$ is harmonic in D_w and zero at infinity, the minimum principle and (3.9) give that $U^{\nu}(z) = U^{\mu_w}(z)$ for $z \in D_w$, and therefore,

$$U^{\nu}(z) = U^{\mu_w}(z), \qquad \text{q.e.} \quad z \in \partial D_w.$$

Consequently, equality holds in every inequality in (3.9) for q.e. $z \in \partial D_w$. Then it follows that $\liminf_{n \in \Lambda} c_n = 0$. Since this holds for every subsequence $\Lambda \subset \mathbf{N}$ for which $\{\nu_n\}_{n \in \Lambda}$ converges, we obtain

$$\lim_{n \to \infty} c_n = 0. \tag{3.10}$$

Since for a closed set $A \subset D_w$ there exists a constant $C > 0$ such that $g_{D_w}(z; \infty) \geqslant C$ for $z \in A$, it follows from (3.8) and (3.10) that

$$\lim_{n \to \infty} \nu(p_n)(A) = 0.$$

This proves (3.5).

To prove the rest of the theorem, let ν^* be the weak* limit of a subsequence of $\{\nu(p_n)\}$; say Λ is a subsequence of the natural numbers such that $\nu(p_n) \to \nu^*$ as $n \to \infty$, $n \in \Lambda$. Having (3.5), we see that ν^* is supported on $Pc(S_w)$. Define

$$A := \{ z \in D_w : \operatorname{dist}(z, S_w) \geqslant 1 \}.$$

Let $\zeta_{j,n}$, $j=1, ..., n$, be the zeros of p_n counted according to multiplicity, and put

$$r_n(z) := \prod_{\zeta_{j,n} \in A} (z-\zeta_{j,n}), \qquad q_n(z) := \frac{p_n(z)}{r_n(z)} = \prod_{\zeta_{j,n} \notin A} (z-\zeta_{j,n}).$$

Then, because of (3.5),

$$\deg q_n = n(1-\delta_n), \qquad \delta_n \to 0, \tag{3.11}$$

and the sequence $\{v(q_n)\}_{n \in \Lambda}$ converges to v^* in the weak* sense. Since the measures $v(q_n)$ are supported on a fixed compact set, the lower envelope theorem can be applied. It gives

$$U^{v^*}(z) = \liminf_{n \to \infty,\, n \in \Lambda} U^{v(q_n)}(z), \qquad \text{q.e. } z \in \mathbf{C}. \tag{3.12}$$

Next, since $r_n(z) \geqslant 1$ for $z \in S_w$, we have for $z \in S_w$

$$U^{v(p_n)}(z) = (1-\delta_n)\, U^{v(q_n)}(z) - \delta_n \log |r_n(z)| \leqslant (1-\delta_n)\, U^{v(q_n)}(z);$$

hence, by (3.11), (3.12),

$$\liminf_{n \to \infty,\, n \in \Lambda} U^{v(p_n)}(z) \leqslant \liminf_{n \to \infty,\, n \in \Lambda} [(1-\delta_n)\, U^{v(q_n)}(z)]$$
$$= U^{v^*}(z), \qquad \text{q.e. } z \in S_w.$$

Combining this with (3.6), we obtain

$$U^{\mu_w}(z) \leqslant U^{v^*}(z), \qquad \text{q.e. } z \in \partial D_w.$$

In the same way as before, cf. (3.9), this implies equality for q.e. $z \in \partial D_w$. Now the equality of the balayages of v^* and μ_w onto ∂D_w follows from the uniqueness of balayage. This completes the proof of Theorem 5. ∎

4. PROOF OF THEOREM 1

We start with a lemma which will also be useful for the proof of Theorem 2.

LEMMA 6. *Let μ_0 and μ_1 be measures satisfying Assumptions A and B. Let π_n be the sequence of monic orthogonal polynomials with respect to* (1.1). *Then we have*

$$\limsup_{n \to \infty} \|\pi_n\|_{\Sigma_0}^{1/n} \leqslant \operatorname{cap}(\Sigma) \tag{4.1}$$

and

$$\limsup_{n\to\infty} \|\pi_n'\|_{\Sigma}^{1/n} \leqslant \operatorname{cap}(\Sigma). \tag{4.2}$$

Proof. Let $\|\cdot\|_H$ denote the norm associated with the inner product (1.1),

$$\|f\|_H^2 = \|f\|_{L^2(\mu_0)}^2 + \|f'\|_{L^2(\mu_1)}^2.$$

We first prove that

$$\limsup_{n\to\infty} \|\pi_n\|_H^{1/n} \leqslant \operatorname{cap}(\Sigma). \tag{4.3}$$

Let T_n be the monic Chebyshev polynomial of degree n for Σ. That is, $\|T_n\|_\Sigma \leqslant \|P_n\|_\Sigma$ for all monic polynomials P_n of degree n. It is well known that

$$\lim_{n\to\infty} \|T_n\|_\Sigma^{1/n} = \operatorname{cap}(\Sigma). \tag{4.4}$$

From the regularity of Σ_1 (see Assumption A) it is easy to see (using the continuity of the Green function, the Bernstein–Walsh lemma and Cauchy's formula) that the Markov constants for Σ_1 have subexponential growth. This means that there exist constants M_n with $\lim_{n\to\infty} M_n^{1/n} = 1$ such that

$$\|P_n'\|_{\Sigma_1} \leqslant M_n \|P_n\|_{\Sigma_1}, \qquad \deg P_n \leqslant n. \tag{4.5}$$

Then, for certain constants c_1, c_2,

$$\begin{aligned} \|T_n\|_H^2 = \|T_n\|_{L_2(\mu_0)}^2 + \|T_n'\|_{L_2(\mu_1)}^2 &\leqslant c_1 \|T_n\|_{\Sigma_0}^2 + c_2 \|T_n'\|_{\Sigma_1}^2 \\ &\leqslant c_1 \|T_n\|_{\Sigma_0}^2 + c_2 M_n^2 \|T_n\|_{\Sigma_1}^2 \leqslant (c_1 + c_2 M_n^2) \|T_n\|_\Sigma^2. \end{aligned} \tag{4.6}$$

Using (4.4), (4.6), and $M_n^{1/n} \to 1$, we find

$$\limsup_{n\to\infty} \|T_n\|_H^{1/n} \leqslant \operatorname{cap}(\Sigma).$$

Since π_n minimizes the Sobolev norm among all monic polynomials of degree n, we have $\|\pi_n\|_H \leqslant \|T_n\|_H$ for all n, and (4.3) follows.

Now, because $\mu_0 \in \mathbf{Reg}$, we have by (1.6),

$$\lim_{n\to\infty} \left(\frac{\|\pi_n\|_{\Sigma_0}}{\|\pi_n\|_{L^2(\mu_0)}} \right)^{1/n} = 1. \tag{4.7}$$

Since $\|\pi_n\|_{L^2(\mu_0)} \leqslant \|\pi_n\|_H$, we get (4.1) from (4.3) and (4.7).

Next, using the regularity of Σ_0, we find that the Markov constants for Σ_0 grow subexponentially. Thus,

$$\limsup_{n \to \infty} \left(\frac{\|\pi_n'\|_{\Sigma_0}}{\|\pi_n\|_{\Sigma_0}} \right)^{1/n} \leqslant 1.$$

Hence, from (4.1),

$$\limsup_{n \to \infty} \|\pi_n'\|_{\Sigma_0}^{1/n} \leqslant \limsup_{n \to \infty} \|\pi_n\|_{\Sigma_0}^{1/n} \leqslant \operatorname{cap}(\Sigma). \tag{4.8}$$

Further, we get from $\mu_1 \in \mathbf{Reg}$ and (1.6)

$$\limsup_{n \to \infty} \left(\frac{\|\pi_n'\|_{\Sigma_1}}{\|\pi_n'\|_{L^2(\mu_1)}} \right)^{1/n} \leqslant 1. \tag{4.9}$$

Since $\|\pi_n'\|_{L^2(\mu_1)} \leqslant \|\pi_n\|_H$, (4.3) and (4.9) give

$$\limsup_{n \to \infty} \|\pi_n'\|_{\Sigma_1}^{1/n} \leqslant \operatorname{cap}(\Sigma). \tag{4.10}$$

Combining (4.8) and (4.10), we obtain (4.2). ▮

Remark. Actually, we have equality in (4.1) and (4.2), and we can replace the lim sup's by lim's, but this will not be used in the proof. It is straightforward to see that equality holds in (4.2); cf. the discussion after Lemma 4. Since Σ_0 has positive capacity, it then also follows that

$$\lim_{n \to \infty} \|\pi_n'\|_{\Sigma_0}^{1/n} = \operatorname{cap}(\Sigma).$$

Using (4.8), we obtain equality in (4.1) as well.

Proof of Theorem 1. The theorem follows immediately from Lemma 4 and (4.2). ▮

5. PROOF OF THEOREM 2

Recall the definitions of Ω, V, V_r, and K from Section 1. The significance of the set V is described in the following lemma.

LEMMA 7. *Let $z \in \mathbf{C}$. Then $z \notin V$ if and only if for every $r > g_\Omega(z; \infty)$, there is a differentiable path $\gamma: [0, 1] \to \mathbf{C}$ such that*

(a) $\gamma(0) \in \Sigma_0$,

(b) $\gamma(1) = z$,

(c) $g_\Omega(\gamma(t); \infty) < r$ *for all* $t \in [0, 1]$.

Proof. If $z \in V$, then $z \in V_r$ for some $r > g_\Omega(z;\infty)$. From the definition of V_r it follows that the connected component of $\{\zeta: g_\Omega(\zeta;\infty) < r\}$ containing z does not contain a point of Σ_0. Hence there is no path satisfying (a), (b), and (c).

On the other hand, if $z \notin V$ and $r > g_\Omega(z;\infty)$, then $z \notin V_r$. Thus the connected component of $\{\zeta: g_\Omega(\zeta;\infty) < r\}$ does contain a point of Σ. Consequently, there is a path satisfying (a), (b), and (c). ∎

This allows us to estimate $|\pi_n(z)|$ for z outside V.

LEMMA 8. *For every* $z \in \mathbf{C} \setminus V$,

$$\limsup_{n\to\infty} |\pi_n(z)|^{1/n} \leqslant \operatorname{cap}(\Sigma)\, e^{g_\Omega(z;\infty)}. \tag{5.1}$$

Proof. Let $z \in \mathbf{C}\setminus V$ and $r > g_\Omega(z;\infty)$. By Lemma 7 there is a differentiable path $\gamma: [0,1] \to \mathbf{C}$ satisfying (a), (b), and (c) of Lemma 7. By the Bernstein–Walsh lemma we have

$$|\pi_n'(\zeta)| \leqslant \|\pi_n'\|_\Sigma\, e^{n g_\Omega(\zeta;\infty)}, \qquad \zeta \in \mathbf{C}.$$

Using this and the properties of γ, we find

$$|\pi_n(z)| \leqslant |\pi_n(\gamma(0))| + \left| \int_\gamma \pi_n'(\zeta)\, d\zeta \right| \leqslant \|\pi_n\|_{\Sigma_0} + L(\gamma)\, \|\pi_n'\|_\Sigma\, e^{nr},$$

where $L(\gamma)$ denotes the length of γ. Then, by (4.1) and (4.2),

$$\limsup_{n\to\infty} |\pi_n(z)|^{1/n} \leqslant \operatorname{cap}(\Sigma)\, e^{r}.$$

Since $r > g_\Omega(z;\infty)$ can be chosen arbitrarily close to $g_\Omega(z;\infty)$, (5.1) follows. ∎

Proof of Theorem 2. Define

$$w(z) := \exp(-g_\Omega(z;\infty)), \qquad z \in K.$$

Let $\hat{\omega}$ be the balayage of ω_Σ onto K. Since $\Sigma \subset Pc(K)$, we have

$$U^{\hat{\omega}}(z) = U^{\omega_\Sigma}(z), \qquad z \in K.$$

We also have

$$U^{\omega_\Sigma}(z) + g_\Omega(z;\infty) = -\log \operatorname{cap}(\Sigma), \qquad z \in \mathbf{C},$$

so that

$$U^{\hat{\omega}}(z) - \log w(z) = -\log \operatorname{cap}(\Sigma), \qquad z \in K.$$

Thus, by (3.3),

$$\mu_w = \hat{\omega}, \qquad F_w = -\log \operatorname{cap}(\Sigma).$$

Because of (5.1) we can apply Theorem 5, and Theorem 2 follows. ∎

6. COMPUTATIONAL METHODS

There are two general procedures for calculating Sobolev orthogonal polynomials: the modified Chebyshev algorithm [6, Section 2] and the Stieltjes algorithm [6, Section 4]. Both generate the coefficients β_j^k in the recursion

$$\pi_{k+1}(t) = t\pi_k(t) - \sum_{j=0}^{k} \beta_j^k \pi_{k-j}(t), \qquad k = 0, 1, 2, ..., \tag{6.1}$$

for the respective polynomials π_k. Being interested in the polynomials up to (and including) degree n, we need the coefficients $\{\beta_j^k\}_{0 \leqslant j \leqslant k}$ for $k = 0, 1, ..., n-1$.

6.1. *Modified Chebyshev Algorithm*

This computes the desired coefficients $\{\beta_j^k\}$ from "modified moments"

$$\begin{aligned} v_j^{(0)} &= \int p_j(t)\, d\mu_0(t), \qquad 0 \leqslant j \leqslant 2n-1, \\ v_j^{(1)} &= \int p_j(t)\, d\mu_1(t), \qquad 0 \leqslant j \leqslant 2n-2 \quad (\text{if } n \geqslant 2), \end{aligned} \tag{6.2}$$

where $\{p_j\}$ is a given set of polynomials, with p_j monic of degree j. "Ordinary moments" correspond to $p_j(t) = t^j$, but are numerically unsatisfactory. A better choice are modified moments corresponding to a set $\{p_j\}$ of orthogonal polynomials, $p_j(\cdot) = p_j(\,\cdot\,; \lambda)$, relative to some suitable measure λ on $\mathbf{R}$. These are known to satisfy a three-term recurrence relation,

$$\begin{aligned} p_{k+1}(t) &= (t - a_k)\, p_k(t) - b_k p_{k-1}(t), \qquad k = 0, 1, 2, ..., \\ p_0(t) &= 1, \qquad p_{-1}(t) = 0, \end{aligned} \tag{6.3}$$

with coefficients $a_k = a_k(\lambda)$, $b_k = b_k(\lambda)$ depending on λ. We need the coefficients $\{a_j\}$, $\{b_j\}$ for $0 \leqslant j \leqslant 2n-2$.

In the context of the Sobolev orthogonal polynomials of Section 2, a natural choice of λ, and one that was found to work well, is $\lambda = \lambda_+ + \lambda_-$. By the orthogonality of the p_j we then have

$$\int_{-1}^{-1/2} p_j(t)\, d\lambda_-(t) + \int_{1/2}^{1} p_j(t)\, d\lambda_+(t) = 0, \qquad j \geqslant 1,$$

so that

$$\int_{-1}^{-1/2} p_j(t)\, d\lambda_-(t) = -\int_{1/2}^{1} p_j(t)\, d\lambda_+(t). \tag{6.4}$$

Since, by symmetry, $p_j(-t) = (-1)^j\, p_j(t)$, the change of variables $t = -\tau$ in (6.4) yields

$$\int_{1/2}^{1} p_j(t)\, d\lambda_+(t) = 0 \qquad \text{if } j \text{ is even } \geqslant 2. \tag{6.5}$$

Let

$$I_j = \int_{1/2}^{1} p_j(t)\, d\lambda_+(t), \qquad 0 \leqslant j \leqslant 2n-1, \tag{6.6}$$

so that $I_j = 0$ if $j \geqslant 2$ is even. We then have, in Case A,

$$v_j^{(0)} = v_j^{(1)} = 2\delta_{j,0} I_0, \qquad j = 0, 1, 2, ..., \tag{6.7}$$

where $\delta_{j,0}$ is the Kronecker delta. Similarly, in Case B,

$$v_j^{(0)} = 2\delta_{j,0} I_0, \qquad v_j^{(1)} = \begin{cases} I_0, & j = 0, \\ -I_j, & j \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{6.8}$$

in Case C:

$$v_j^{(0)} = \left.\begin{cases} I_j, & j = 0 \text{ or } j \text{ odd}, \\ 0, & \text{otherwise}, \end{cases}\right\}, \qquad v_j^{(1)} = 2\delta_{j,0} I_0, \tag{6.9}$$

and in Case D:

$$v_j^{(0)} = \left.\begin{cases} I_j, & j = 0 \text{ or } j \text{ odd}, \\ 0, & \text{otherwise}, \end{cases}\right\}, \quad v_0^{(1)} = I_0, \quad v_j^{(1)} = -v_j^{(0)}, \quad j \geqslant 1. \tag{6.10}$$

In Sections 2.1–2.4 we have that λ_+ and λ_- are Lebesgue measure supported on $[\frac{1}{2}, 1]$ and $[-1, -\frac{1}{2}]$, respectively. Here, $I_0 = \frac{1}{2}$. The coefficients $a_j(\lambda)$, $b_j(\lambda)$ in (6.3) can be computed very accurately by known procedures of Stieltjes or Lanczos type (cf. [3, Example 4.7; 5, Section 4.3]), whereupon the integrals I_j in (6.6) can be computed (exactly) by (6.3) and n-point Gauss–Legendre quadrature.

In Section 2.5, λ_+ and λ_- are equal to the measure $|t|\,(t^2-\frac{1}{4})^{-1/2} \times (1-t^2)^{-1/2}$ supported on $[\frac{1}{2}, 1]$ and $[-1, -\frac{1}{2}]$, respectively. Here, $I_0 = \frac{1}{2}\pi$. The coefficients $a_j(\lambda)$, $b_j(\lambda)$ are known explicitly (cf. [4, Section 5.1]):

$$\begin{aligned} &a_j = 0, && 0 \leqslant j \leqslant 2n-2, \\ &b_0 = \pi, && b_1 = \tfrac{5}{8}, \\ &b_j = \frac{1}{16} \begin{cases} 9\,\dfrac{1+3^{j-2}}{1+3^j}, & j \text{ even}, \\ \dfrac{1+3^{j+1}}{1+3^{j-1}}, & j \text{ odd}, \end{cases} && j = 2, 3, ..., 2n-2. \end{aligned} \tag{6.11}$$

The integrals I_j can no longer be computed exactly by numerical quadrature, but can be approximated by N-point Gauss–Chebyshev quadrature with N sufficiently large. Indeed, if in

$$I_j = \int_{1/2}^{1} p_j(t)\, t(t^2 - \tfrac{1}{4})^{-1/2}\, (1-t^2)^{-1/2}\, dt$$

one makes the change of variables $t^2 = (1+3s)/4$, one gets

$$I_j = \tfrac{1}{2} \int_0^1 p_j(\tfrac{1}{2}\sqrt{1+3s})\, s^{-1/2}(1-s)^{-1/2}\, ds,$$

or, transforming to the interval $[-1, 1]$,

$$I_j = \frac{1}{2} \int_{-1}^{1} p_j\left(\frac{1}{2\sqrt{2}} \sqrt{5+3x}\right) (1-x^2)^{-1/2}\, dx. \tag{6.12}$$

Gauss–Chebyshev quadrature applied to the integral in (6.12) converges fast.

6.2. *Stieltjes Algorithm*

Here the coefficients $\{\beta_j^k\}$ are computed as Fourier–Sobolev coefficients

$$\beta_j^k = \frac{(t\pi_k, \pi_{k-j})_H}{\|\pi_{k-j}\|_H^2}, \qquad j = 0, 1, ..., k, \tag{6.13}$$

where appropriate quadrature rules are used to compute the inner products in (6.13). The coefficients β_j^k and polynomials π_m intervening in (6.13) are computed simultaneously, the polynomials recursively by (6.1) using the coefficients β_j^k already obtained. The choice of quadrature rules is particularly simple in the case of Lebesgue measures. Indeed, for $k \leqslant n-1$, the integrands in (6.13) are polynomials of degree $\leqslant 2n-1$, so that n-point Gauss–Legendre rules on the respective intervals $[-1, -\frac{1}{2}]$ and $[\frac{1}{2}, 1]$ will do the job. In the other example, one has to integrate numerically as described above in connection with I_j.

6.3. *Zeros*

The zeros of π_n (including the complex ones, if any) can be conveniently computed as eigenvalues of the Hessenberg matrix (cf. [6, Section 1])

$$B_n = \begin{bmatrix} \beta_0^0 & \beta_1^1 & \beta_2^2 & \cdots & \beta_{n-2}^{n-2} & \beta_{n-1}^{n-1} \\ 1 & \beta_0^1 & \beta_1^2 & \cdots & \beta_{n-3}^{n-2} & \beta_{n-2}^{n-1} \\ 0 & 1 & \beta_0^2 & \cdots & \beta_{n-4}^{n-2} & \beta_{n-3}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_0^{n-2} & \beta_1^{n-1} \\ 0 & 0 & 0 & \cdots & 1 & \beta_0^{n-1} \end{bmatrix}. \tag{6.14}$$

To compute all real zeros of π_n and π'_n, we scanned a suitable interval (typically, $[-1.6, 1.6]$) for sign changes in π_n and π'_n and used the midpoints of the smallest intervals found on which π_n (resp. π'_n) changes sign as initial approximations to Newton's method.

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